SOLID-PHASE COMBUSTION IN A PLANE STRESS STATE.

1. STATIONARY COMBUSTION WAVE

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A model that describes propagation of the conversion front in a generalized plane stress state typical for technological conditions of coating synthesis on a substrate, with allowance for the coupled character of heat transfer and deformation without external mechanical loading, is proposed. A stationary solution is obtained in the approximation of a narrow combustion front. Ranges of model parameters where the temperature of reaction products and the components of stress and strain tensors behave differently are identified.

Key words: synthesis of coatings on a substrate, solid-phase combustion, plane stress state.

Introduction. The combustion process in a global sense is layer-by-layer propagation of the zone of exothermal conversion. This definition is valid for gases, liquids, and solids. If the characteristic size of the chemical reaction area is large, as compared with the combustion zone size, then the thermal aspect of the problem can be separated from the hydrodynamic aspect [1]. For the same reason, the reaction zone in theoretical research is replaced by a discontinuity surface with the temperature remaining unchanged in passing through this surface. The discontinuity in the heat flux is conditioned by heat release in the course of the reaction. The propagating reaction front can be unstable to small perturbations leading to front curving, changes in velocity, etc. This problem, which was actively discussed in the 1960–1980s in view of investigations of gas-phase, heterogeneous, and solid-phase reactions, is still important now owing to the development of new technologies of obtaining materials, applying coatings, or surface processing.

Many technological processes (or their stages) of obtaining materials and applying coatings are accompanied by physical and chemical transformations of various types or are based on using the energy of chemical reactions. For this reason, mathematical models of high-temperature technological processes are similar to models used in the combustion theory. Real technological processes, however, are much more complicated than the processes described by the thermal theory of combustion. Thus, typical features of solid-phase combustion are proceeding of the reaction in the solid (condensed) phase and the absence of gasification. This process, which was called gasless combustion [2], is the basis, for instance, of the technology of coating synthesis on a substrate, initiated and controlled by a scanning electron beam. It turned out that the regime of conversion on the substrate is not always stable and depends on a number of factors: properties and thickness of the substrate, character of the external action, and composition and structure of the reagent [3]. In any case, synthesis of new materials is accompanied by various interrelated physical and chemical processes affecting the conversion regimes directly or indirectly. If transformations occur in the solid phase, an important role belongs to various mechanical stresses and strains: micro- and macrostresses, thermal concentration and structural, elastic and viscous, etc. (see, e.g., [4, 5]).

One of the first theoretical studies of solid-phase combustion was described in [6]. The characteristics of the stationary combustion regime can be obtained by solving the stationary system of equations

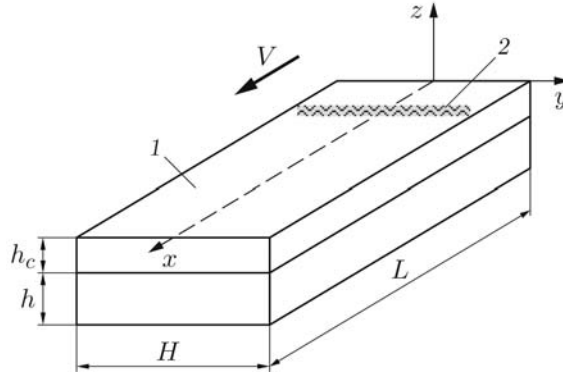$$c\rho V_{n0} \frac{dT}{dx} = \lambda_T \frac{d^2T}{dx^2} + Q_0 \Phi(T, \eta),$$

Fig. 1. Sketch of the problem: 1) coating; 2) reaction zone.

$$V_{n0} \frac{d\eta}{dx} = \Phi(T, \eta), \tag{1}$$

$$x \to -\infty: \quad T = T_0, \quad \eta = 0, \qquad x \to +\infty: \quad T = T_{b0}, \quad \eta = 1,$$

where T is the temperature, η is the degree of conversion, x is the spatial coordinate, V_{n0} is the velocity of the stationary front, c, ρ, and λ_T are the specific heat, density, and thermal conductivity, T_0 is the initial temperature or the temperature of the reagents, Q_0 is the thermal effect of the overall exothermal reaction "reagent–product," the function $\Phi(T, \eta)$ is taken in the form

$$\Phi(T, \eta) = k_0 \varphi_1(\eta) \varphi_2(T), \tag{2}$$

k_0 is the pre-exponent, φ_1 is the kinetic function, $\varphi_2(T) = \exp\left(-E_a/(RT)\right)$, E_a is the activation energy of the chemical reaction, R is the universal gas constant, and T_{b0} is the temperature of the reaction products in the purely thermal model of solid-phase combustion determined from the first integral of system (1):

$$T_{b0} = T_0 + Q_0/(c\rho). \tag{3}$$

In the approximation of a narrow reaction front, system (1) yields a problem with ignored volume sources of heat in the equations. The conditions of temperature and heat-flux continuity

$$T_1 = T_2, \qquad \left.\frac{dT_1}{dx}\right|_{x=x_0} - \left.\frac{dT_2}{dx}\right|_{x=x_0} = \frac{Q_0}{\lambda_T} V_{n0} \tag{4}$$

are valid on the interface between the reagents and the reaction products (indicated by the subscripts 1 and 2, respectively). Thus, the discontinuity in the heat flux is caused by heat release in the course of the chemical reaction.

If this approach is used, the solution of the problem is the Michelson profile of temperature of the reagents:

$$T = T_0 + (T_{b0} - T_0) \exp\left(xV_{n0}/\varkappa_T\right) \qquad (x < 0)$$

[$\varkappa_T = \lambda_T/(c\rho)$ is the thermal diffusivity]. The velocity of motion of the stationary front depends on the form of the kinetic function [6].

Some authors studied the thermal stability of the front to small one-dimensional and two-dimensional perturbations in solid-phase combustion [7, 8]. The effect of thermal expansion and mobility of the medium on propagation of the combustion reaction in the solid phase has not been analyzed for a long time. Nevertheless, the mutual influence of the heat-transfer and deformation processes may be one of the reasons for the formation of various combustion modes, including fast (solid-phase detonation) and slow (solid-phase combustion) regimes. The author of the present paper reviewed publications on this topic in [9]. The possibility of solid-phase detonation was demonstrated in [10] through thermodynamic calculations, and the existence of this phenomenon was actually validated in [11, 12]. The problem of stability of stationary regimes to thermomechanical perturbations in a coupled formulation was first posed in [13].

The objective of the present work is to study stability of the solid-phase conversion front to small one-dimensional perturbations during application of a coating onto a substrate.

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General Formulation of the Problem. Let the reaction front move over the coating in the x direction (Fig. 1). Such a situation can occur is, for instance, the reaction is initiated by an electron beam unfolded into a line [3] and the heat losses on the end faces of the plate are negligibly small. Let us also assume that the reaction is described by an overall reaction scheme, the total thickness of the plate is substantially smaller than its width and length,

$$h + h_s \ll H, \qquad h + h_s \ll L, \tag{5}$$

and there is an ideal contact between the coating and the substrate.

Neglecting the heat losses due to thermal radiation, we write the thermokinetic part of the problem as

$$c_\varepsilon \rho \frac{\partial T}{\partial t} = \lambda_T \, \Delta T + Q_0 \frac{\partial \eta}{\partial t} - 3K\alpha_T T \frac{\partial \varepsilon_{kk}}{\partial t}; \tag{6}$$

$$\frac{\partial \eta}{\partial t} = k_0 \varphi_1(\eta)\varphi_2(T, \Pi),$$

$$T = T_0, \quad \eta = 0, \qquad \nabla T = 0, \quad \eta = 1.$$

Here, c_ε is the specific heat at constant strains, α_T is the temperature coefficient of linear expansion, K is the isothermal compression modulus, and $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ is the first invariant of the strain tensor. In this case, in contrast to Eq. (2), the reaction rate Φ depends on the work of mechanical stresses Π: $\Phi = \Phi(T, \eta, \Pi)$.

The reaction proceeds only in the coating; the coating and substrate materials have different properties. For this reason, it is necessary to solve an adjoint problem of heat conduction in the general case. By virtue of condition (5), however, Eq. (6) can be integrated over the entire sample thickness:

$$c_{\text{eff}}\rho_{\text{eff}} \frac{\partial \tilde{T}}{\partial t} = \lambda_{T,\text{eff}} \left(\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} \right) + Q_0 \frac{\partial \tilde{\eta}}{\partial t} - 3(K\alpha_T)_{\text{eff}} \tilde{T} \frac{\partial \tilde{\varepsilon}_{kk}}{\partial t}. \tag{7}$$

Here, $c_{\text{eff}}\rho_{\text{eff}} = c_c\rho_c + c\rho h/h_c$, $\lambda_{T,\text{eff}} = \lambda_{T,c} + \lambda_T h/h_c$, and $(K\alpha_T)_{\text{eff}} = K_s\alpha_{T,s} + K\alpha_T h/h_c$; the subscript c refers to the coating parameters; the tilde indicate averaging over the thickness. In the case of a plane reaction front, we obtain a problem in a one-dimensional formulation.

To find the components of the strain tensor ε_{ij} and the work Π, we have to solve the problem of mechanical equilibrium. As the thermal and mechanical processes are characterized by substantially different scales and rates, a quasi-static problem is usually solved. As the dynamic effects can lead to the emergence of new conversion modes, including the regime of solid-phase detonation [5, 9], however, we take into account the inertial forces in formulating the second part of the problem. Assuming that strains are small, we write the equilibrium equations with allowance for the inertial forces in the form

$$\nabla \cdot \sigma = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2},$$

where σ is the stress tensor with the components $\sigma_{ij} = f(\varepsilon_{kl}, T, \eta)$ and $\boldsymbol{u}$ is the displacement vector. The generalized theory of thermal and mass elasticity includes the equality

$$d\sigma_{ij} = 2\mu \, d\varepsilon_{ij} + \delta_{ij}(\lambda \, d\varepsilon_{kk} - K \, dw), \tag{8}$$

where $dw = 3[\alpha_T dT + (\alpha_2 - \alpha_1) \, d\eta]$ if there is only one reaction, λ and μ are the Lamé coefficients, $K = \lambda + 2\mu/3$, δ_{ij} is the Kronecker delta, and α_2 and α_1 are the coefficients of concentration expansion of the product and reagent, respectively. The difference $\alpha_2 - \alpha_1$ shows whether the reaction proceeds with increasing or decreasing volume.

Equation (8) yields the relation

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^T + d\varepsilon_{ij}^{ch},$$

where $d\varepsilon_{ij}^e = d\sigma_{ij}/(2\mu) - \delta_{ij}\lambda \, d\sigma_{kk}/(6K\mu)$ are reversible deformations induced by elastic stresses, $d\varepsilon_{ij}^T = \delta_{ij}\alpha_T \, dT$ are the thermal strains, and $d\varepsilon_{ij}^{ch} = \delta_{ij}(\alpha_2 - \alpha_1) \, d\eta$ are the concentration or (in this case) chemical deformations. Viscous stresses are ignored in this model. There is no external mechanical loading.

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In the case of a plane stress state, in view of conditions (5), the problem can be simplified. As a result, the stress and strain tensors take the form [14]

$$
\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad
\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}.
$$

Then, in the case of small strains, we obtain

$$
\sigma_{yy} = \frac{2\mu}{\lambda + 2\mu} [\lambda \varepsilon_{xx} + 2(\lambda + \mu)\varepsilon_{yy} - Kw],
$$

$$
\sigma_{xx} = \frac{2\mu}{\lambda + 2\mu} [\lambda \varepsilon_{yy} + 2(\lambda + \mu)\varepsilon_{xx} - Kw], \tag{9}
$$

$$
\varepsilon_{zz} = \frac{Kw}{\lambda + 2\mu} - \frac{\lambda}{\lambda + 2\mu}(\varepsilon_{xx} + \varepsilon_{yy}).
$$

Differentiating the first two equations in system (9) twice with respect to y and x and taking into account that

$$
\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),
$$

we obtain the equations of motion with respect to the strains ε_{xx} and ε_{yy} for the coating and the substrate:

$$
\frac{2\mu}{\lambda + 2\mu}\Big(\lambda \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} + 2(\lambda + \mu)\frac{\partial^2 \varepsilon_{xx}}{\partial x^2} - K\frac{\partial^2 w}{\partial x^2}\Big) + \mu\Big(\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}\Big) = \rho \frac{\partial^2 \varepsilon_{xx}}{\partial t^2}; \tag{10}
$$

$$
\frac{2\mu}{\lambda + 2\mu}\Big(\lambda \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + 2(\lambda + \mu)\frac{\partial^2 \varepsilon_{yy}}{\partial y^2} - K\frac{\partial^2 w}{\partial y^2}\Big) + \mu\Big(\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}\Big) = \rho \frac{\partial^2 \varepsilon_{yy}}{\partial t^2}. \tag{11}
$$

As the plate properties are inhomogeneous over its thickness, we obtain the following relations in the general case: $\sigma_{yy} = \sigma_{yy}(x, y, z)$, $\sigma_{xx} = \sigma_{xx}(x, y, z)$, etc. As there is an ideal contact between the coating and the substrate and the plate is thin, we can integrate Eq. (10) over the plate thickness and obtain

$$
\frac{2\mu_c}{\lambda_c + 2\mu_c} h_c\Big(\lambda_{c,\text{eff}} \frac{\partial^2 \tilde{\varepsilon}_{yy}}{\partial x^2} + 2(\lambda_c + \mu_c)_{\text{eff}}\frac{\partial^2 \tilde{\varepsilon}_{xx}}{\partial x^2} - K_{c,\text{eff}}\frac{\partial^2 \tilde{w}}{\partial x^2}\Big) + \mu_{c,\text{eff}} h_c\Big(\frac{\partial^2 \tilde{\varepsilon}_{xx}}{\partial y^2} + \frac{\partial^2 \tilde{\varepsilon}_{yy}}{\partial x^2}\Big) = (\rho_c h_c + \rho h)\frac{\partial^2 \tilde{\varepsilon}_{xx}}{\partial t^2},
$$

where $\lambda_{c,\text{eff}}$, $K_{c,\text{eff}}$, and $\mu_{c,\text{eff}}$ are the effective properties depending on the relations between the material parameters and their thickness. In what follows, the tilde is omitted.

Equation (11) is integrated in a similar manner.

As in the case of the thermal problem, the problem of reaction front propagation becomes one-dimensional if the reaction front is plane [15, 16].

Obviously, if the coating and the substrate materials have identical properties or $h = 0$, we obtain a simpler problem. In the first case, the presence of the substrate leads to a decrease in effective heat release in the course of the reaction by a factor of $1 + h/h_c$; in the second case, the substrate only promoted retaining of a plane stress state. The simplified problem is analyzed below. In the case considered, the presence of the substrate leads to reduction of the total heat release in the reaction.

The first stage of solving the problem is the analysis of the stationary (undisturbed) pattern.

Stationary Reaction Front. As the reaction front is plane and moves in the x direction, whereas the x and y sizes of the sample are large, then we have

$$
T = T(x, t), \quad \eta = \eta(x, t), \quad \varepsilon_{ij} = \varepsilon_{ij}(x, t), \quad \sigma_{ij} = \sigma_{ij}(x, t)
$$

and

$$
\varepsilon_{xy} = 0, \qquad \sigma_{xy} = 0.
$$

In the front-fitted coordinate system, Eqs. (7), (10), and (11) take the form

$$
\frac{\partial \eta}{\partial t} + V_n \frac{\partial \eta}{\partial x} = k_0 \varphi_1(\eta)\varphi_2(T, \Pi),
$$

$$
c_\varepsilon \rho\Big(\frac{\partial T}{\partial t} + V_n \frac{\partial T}{\partial x}\Big) = \lambda_T \frac{\partial^2 T}{\partial x^2} + Q_{\text{eff}}\Big(\frac{\partial \eta}{\partial t} + V_n \frac{\partial \eta}{\partial x}\Big) - 3K\alpha_T T\Big(\frac{\partial \varepsilon_{kk}}{\partial t} + V_n \frac{\partial \varepsilon_{kk}}{\partial x}\Big),
$$

$$
\frac{2\mu}{\lambda + 2\mu}\Big(\lambda \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} + 2(\lambda + \mu)\frac{\partial^2 \varepsilon_{xx}}{\partial x^2} - K\frac{\partial^2 w}{\partial x^2}\Big) + \mu \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \rho\Big(\frac{\partial^2 \varepsilon_{xx}}{\partial t^2} + 2V_n \frac{\partial^2 \varepsilon_{xx}}{\partial t\, \partial x} + V_n^2 \frac{\partial^2 \varepsilon_{xx}}{\partial x^2}\Big), \tag{12}
$$

$$
\mu \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \rho\Big(\frac{\partial^2 \varepsilon_{yy}}{\partial t^2} + 2V_n \frac{\partial^2 \varepsilon_{yy}}{\partial t\, \partial x} + V_n^2 \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}\Big),
$$

where $Q_{\text{eff}} = Q_0(1 + h/h_c)^{-1}$.

Equations (12) yield the stationary problem

$$V_{n0}\frac{d\eta}{dx} = k_0\varphi_1(\eta)\varphi_2(T,\Pi),$$

$$c\rho V_{n0}\frac{dT}{dx} = \lambda_T\frac{d^2T}{dx^2} + Q_0 V_{n0}\frac{d\eta}{dx} - 3K\alpha_T T V_{n0}\frac{d\varepsilon_{kk}}{dx},$$

$$\frac{2\mu}{\lambda+2\mu}\left(\lambda\frac{d^2\varepsilon_{yy}}{dx^2} + 2(\lambda+\mu)\frac{d^2\varepsilon_{xx}}{dx^2} - K\frac{d^2w}{dx^2}\right) + \mu\frac{d^2\varepsilon_{yy}}{dx^2} = \rho V_{n0}^2\frac{d^2\varepsilon_{xx}}{dx^2}, \qquad (13)$$

$$\mu\frac{d^2\varepsilon_{yy}}{dx^2} = \rho V_{n0}^2\frac{d^2\varepsilon_{yy}}{dx^2},$$

where V_{n0} is the stationary velocity of front motion [as in problem (1)].

From the last equation in system (13), it follows that

$$\frac{d^2\varepsilon_{yy}}{dx^2} = 0$$

or

$$\varepsilon_{yy} = C_1 x + C_2,$$

where the constants of integration can be found from the conditions of zero values of the resultant force and the moment of forces [14]. If the plate sizes are large, the integration constants C_1 and C_2 tend to zero. As a result, from Eqs. (13) and (9), we obtain

$$\varepsilon_{yy} \approx 0, \qquad \varepsilon_{xx} = \frac{2\mu}{\lambda+2\mu}\frac{Kw}{C_0}, \qquad \varepsilon_{zz} = \frac{Kw}{\lambda+2\mu}\frac{2\mu - \rho V_{n0}^2}{C_0},$$

$$\sigma_{xx} = \frac{2\mu Kw}{\lambda+2\mu}\frac{\rho V_{n0}^2}{C_0}, \qquad \sigma_{yy} = -\frac{2\mu Kw}{\lambda+2\mu}\frac{2\mu + \rho V_{n0}^2}{C_0},$$

where

$$C_0 = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} - \rho V_{n0}^2.$$

Therefore, by analogy with the simplest model (1), the problem reduces to solving a system of two equations: the first equation of system (13) and the stationary heat-conduction equation

$$c\rho V_{n0}\frac{dT}{dx} = \lambda_T\frac{d^2T}{dx^2} + Q_{\text{eff}} V_{n0}\frac{d\eta}{dx} - 3K\alpha_T T V_{n0}\frac{4\mu - \rho V_{n0}^2}{C_0}\frac{K}{\lambda+2\mu}\frac{dw}{dx}. \qquad (14)$$

Assuming that $(\alpha_2 - \alpha_1)/\alpha_T \ll 1$, we obtain

$$\frac{dw}{dx} = 3\alpha_T\frac{dT}{dx}.$$

Integrating Eq. (14), similar to [5, 9, 13], we obtain the equation for the temperature of the reaction products. As a result, the stationary problem reduces to the problem analyzed above, but its solution has different areas of existence of various conversion modes because of the plane stress state. Using the method of matched asymptotic expansions, we find the velocity of the stationary front [16]. In this case, the simplified version of the model, similar to the model used in [15], is more suitable here.

For further investigation of stability of the stationary front, we apply linearization to the heat-conduction equation:

$$T\frac{dT}{dx} \approx T_b\left(\frac{dT}{dx}\right)_{T_b} + \left(\frac{dT}{dx}\right)_{T_b}(T - T_b) + T_b\left[\frac{dT}{dx} - \left(\frac{dT}{dx}\right)_{T_b}\right] \approx T_b\frac{dT}{dx}.$$

In this case, the equation for the temperature of the reaction front acquires the form

$$\delta_0\theta_b^2 + (1 + \delta_0\sigma)\theta_b - 1 = 0, \qquad (15)$$

where

$$\delta_0 = \frac{\omega(2\gamma_2 - \alpha^2)}{\gamma_2(1+\gamma_1) - \alpha^2}, \qquad \theta_b = \frac{T_b - T_{b0}}{T_{b0} - T_0}, \qquad \sigma = \frac{T_0}{T_{b0} - T_0}, \qquad \alpha = \frac{V_{n0}^2}{(\lambda+2\mu)/\rho},$$

$$\gamma_1 = \frac{\lambda}{\lambda+2\mu}, \qquad \gamma_2 = \frac{2\mu}{\lambda+2\mu}, \qquad \gamma_1 + \gamma_2 \equiv 1,$$

ω is the coupling coefficient,

$$\omega = \frac{(3K\alpha_T)^2}{\lambda+2\mu}\frac{T_{b0} - T_0}{c\rho},$$

and T_{b0} is the temperature of the reaction products in the purely thermal model (3) with $\theta_b = \theta_{b0} = 1$.
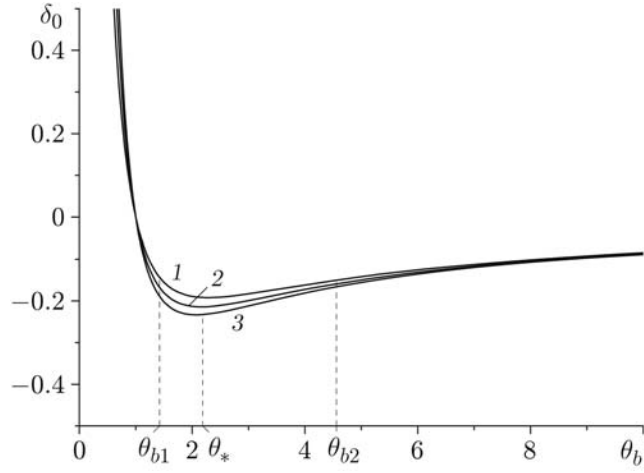
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Fig. 2. Dependence $\delta_0(\theta_b)$ for $\sigma = 0.65$ (1), 0.35 (2), and 0.15 (3).
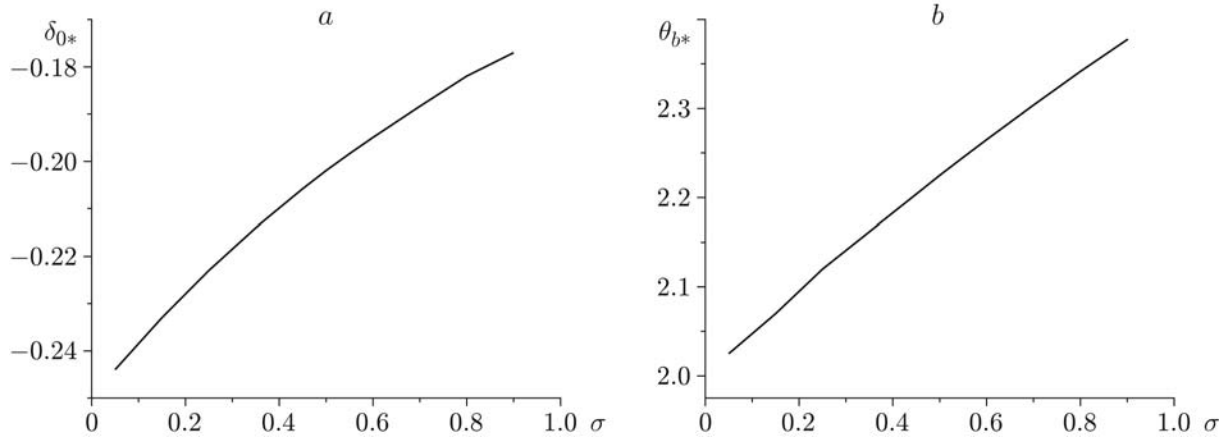

Fig. 3. Critical parameters δ_{0*} (a) and θ_{b*} (b) of existence of the stationary solution versus the temperature difference σ.

We can easily see that Eq. (15) can have two solutions, one solution, or no solutions, depending on the values of the parameters (Fig. 2). The curves $\delta_0(\theta_b)$ in Fig. 2 have a minimum whose value and location depend on σ:

$$\theta_{b*} = 1 + \sqrt{1+\sigma}, \qquad \delta_{0*} = -\frac{\sqrt{1+\sigma}}{2(1+\sigma) + (2+\sigma)\sqrt{1+\sigma}}.$$

With increasing σ, the values of θ_{b*} and δ_{0*} increase (Fig. 3).

At $\delta_0 < \delta_{0*}$, Eq. (15) has no real roots. At $\delta_0 > 0$, there is a unique solution, and $\theta_b < 1$. In the interval $\delta_0 \in (\delta_{0*}, 0)$, Eq. (15) has two positive roots, which apparently determine the combustion-wave structure, as in [5]. The sign of δ_0 depends on other parameters of the model. We can easily show that $\delta_0 > 0$ if $\alpha < \gamma_* = \sqrt{\gamma_2(1+\gamma_1)} = \sqrt{\gamma_2(2-\gamma_2)}$ or $\alpha > \sqrt{2\gamma_2}$. After passing to physical quantities, the equation for the temperature of the reaction products has a unique solution at $\rho V_{n0}^2 < 4\mu$ or $\rho V_{n0}^2 > 4\mu(\lambda + \mu)/(\lambda + 2\mu)$.

In the interval $\alpha \in (\gamma_*, \sqrt{2\gamma_2})$, we have $\delta_0 < 0$. The points $\alpha = \sqrt{2\gamma_2}$ and $\alpha = \gamma_*$ are special points. Thus, if $\alpha = \sqrt{2\gamma_2}$, then the model is uncoupled at $\omega \neq 0$.

In the approximation of a narrow reaction zone, the stationary problem includes the linearized heat-conduction equations (14) with the chemical source of heat being ignored,

$$c\rho V_{n0} \frac{dT_i}{dx} = \lambda_T \frac{d^2 T_i}{dx^2} - T_b V_{n0} \frac{4\mu - \rho V_{n0}^2}{C_0} \frac{(3\alpha_T K)^2}{\lambda + 2\mu} \frac{dT_i}{dx},$$

the boundary conditions (4) with Q_{eff} used instead of Q_0, and conditions similar to conditions in problem (1).

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TABLE 1

Values of θ_{b1} and θ_{b2} for Different Values of α

α	θ_{b1}	θ_{b2}
0.79	1.353	4.54
0.80	1.160	9.01
0.81	1.080	16.57

Let us pass to dimensionless variables:

$$\theta_i = \frac{T_i - T_0}{T_{b0} - T_0}, \qquad X = \frac{V_{n0}x}{\varkappa_T}.$$

As a result, in view of Eq. (3), the problem considered takes the form

$$\frac{d\theta_i}{dX}(1+\delta) = \frac{d^2\theta_i}{dX^2}, \qquad i = 1, 2,$$

$$X = 0: \quad \theta_1 = \theta_2, \quad \frac{d\theta_1}{dX} = \frac{d\theta_2}{dX} + 1,$$

$$X \to -\infty: \quad \theta_1 = 0, \qquad X \to \infty: \quad \theta_2 = \theta_b,$$

where $\delta = \delta_0(\theta_b + \sigma)$.

The solution of the stationary problem indicated further by the superscript 0 exists at $1 + \delta > 0$ and has the form

$$\theta_1^0 = \theta_b \exp[(1+\delta)X], \qquad \theta_2^0 = \theta_b.$$

We can easily derive the expressions

$$e_{xx,i}^0 = \frac{\gamma_2\theta_i^0}{\gamma_2(1+\gamma_1) - \alpha^2}, \qquad e_{zz,i}^0 = \frac{(\gamma_2 - \alpha^2)\theta_i^0}{\gamma_2(1+\gamma_1) - \alpha^2},$$

$$S_{xx,i}^0 = \frac{\gamma_2\alpha^2\theta_i^0}{\gamma_2(1+\gamma_1) - \alpha^2}, \qquad S_{yy,i}^0 = -\frac{\gamma_2(\gamma_2 + \alpha^2)\theta_i^0}{\gamma_2(1+\gamma_1) - \alpha^2},$$

where

$$S_{ij} = \frac{\sigma_{ij}}{\sigma_*}, \quad e_{ij} = \frac{\varepsilon_{ij}}{\varepsilon_*}, \quad \sigma_* = 3K\alpha_T(T_{b0} - T_0), \quad \varepsilon_* = \frac{3K\alpha_T(T_{b0} - T_0)}{\lambda + 2\mu}.$$

Analysis of the Solution Obtained. As the dependence of the temperature of the reaction products is rather complicated and the area of solution existence is subjected to constraints ($\delta_0 > \delta_{0*}$ and $\alpha < \alpha_*$), the stationary solutions are different in different ranges of variation of the parameters γ_2 (or γ_1), α, ω, and σ. In particular, different conversion modes are characterized by different widths of the heating zone, values of θ_b, and signs of the non-zero components of the stress and strain tensors. Thus, for $\omega = 0.1$, $\gamma_1 = 0.65$ ($\gamma_2 = 0.35$), and $\sigma = 0.25$, we have $\sqrt{2\gamma_2} \approx 0.837$ and $\gamma_* = 0.76$. At $\alpha < \sqrt{\gamma_2}$, we have $\delta_0 > 0$; the value of θ_b is unique and weakly depends on α, $e_{xx,1}^0 > 0$, $e_{zz,1}^0 > 0$, $S_{xx}^0 > 0$, and $S_{yy}^0 < 0$. If $\alpha \ll 1$, then we have $S_{xx}^0 \to 0$.

At $\alpha > \sqrt{\gamma_2}$, the strains $e_{zz,1}^0$ change their sign, and the temperature θ_b decreases with increasing α: the closer the value of α to the value of γ_*, the more rapid the decrease in temperature. The distributions of various quantities in a stationary wave for this case are plotted in Fig. 4.

In the interval $\gamma_* < \alpha < \sqrt{2\gamma_2}$, Eq. (15) has two roots θ_b; the smaller temperature θ_{b1} decreases with increasing α, and the greater temperature θ_{b2} increases (see Table 1). For this range of parameters, Fig. 5 shows the distributions of all quantities in the heating zone, which are constructed on the basis of the smaller value of the temperature θ_b. It is seen that the stresses S_{xx}^0 in the heating zone are tensile stresses, in contrast to the case considered above (see Fig. 4). At $\alpha \to \sqrt{2\gamma_2}$, we have $1 + \delta < 0$, and there is no stationary solution. At $\alpha > \sqrt{2\gamma_2}$, there is a unique solution of Eq. (15). In what follows, the temperature θ_b remains practically unchanged with increasing α, and $e_{xx} \to 0$.
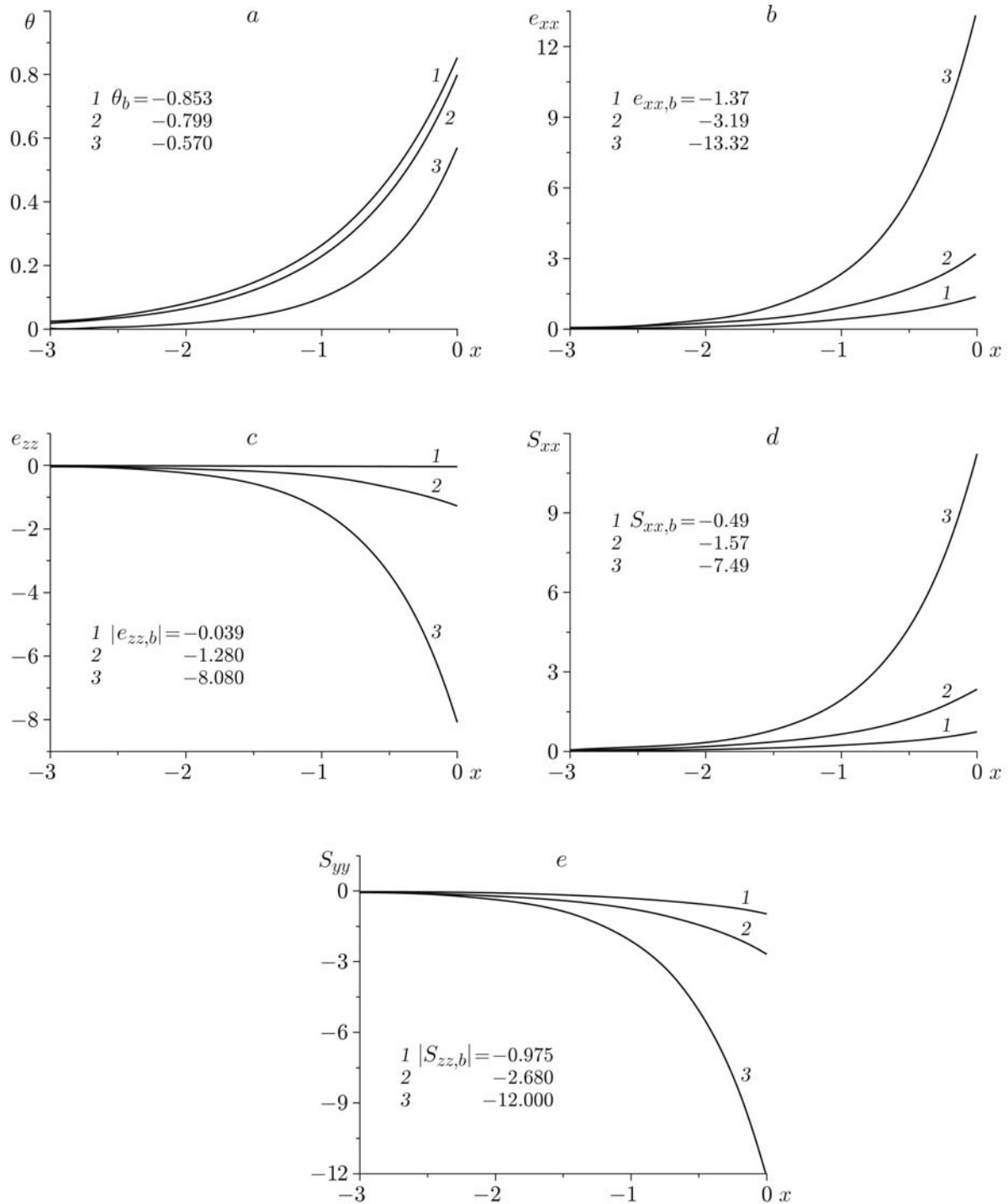
Fig. 4. Distributions of temperature (a) and non-zero components of the strain tensors (b and c) and stress tensors (d and e) in the heating zone at $\sqrt{\gamma_2} < \alpha < \gamma_*$ ($\omega = 0.1$, $\sigma = 0.25$, and $\gamma_2 = 0.35$): $\alpha = -0.6$ (1), -0.7 (2), and -0.75 (3).
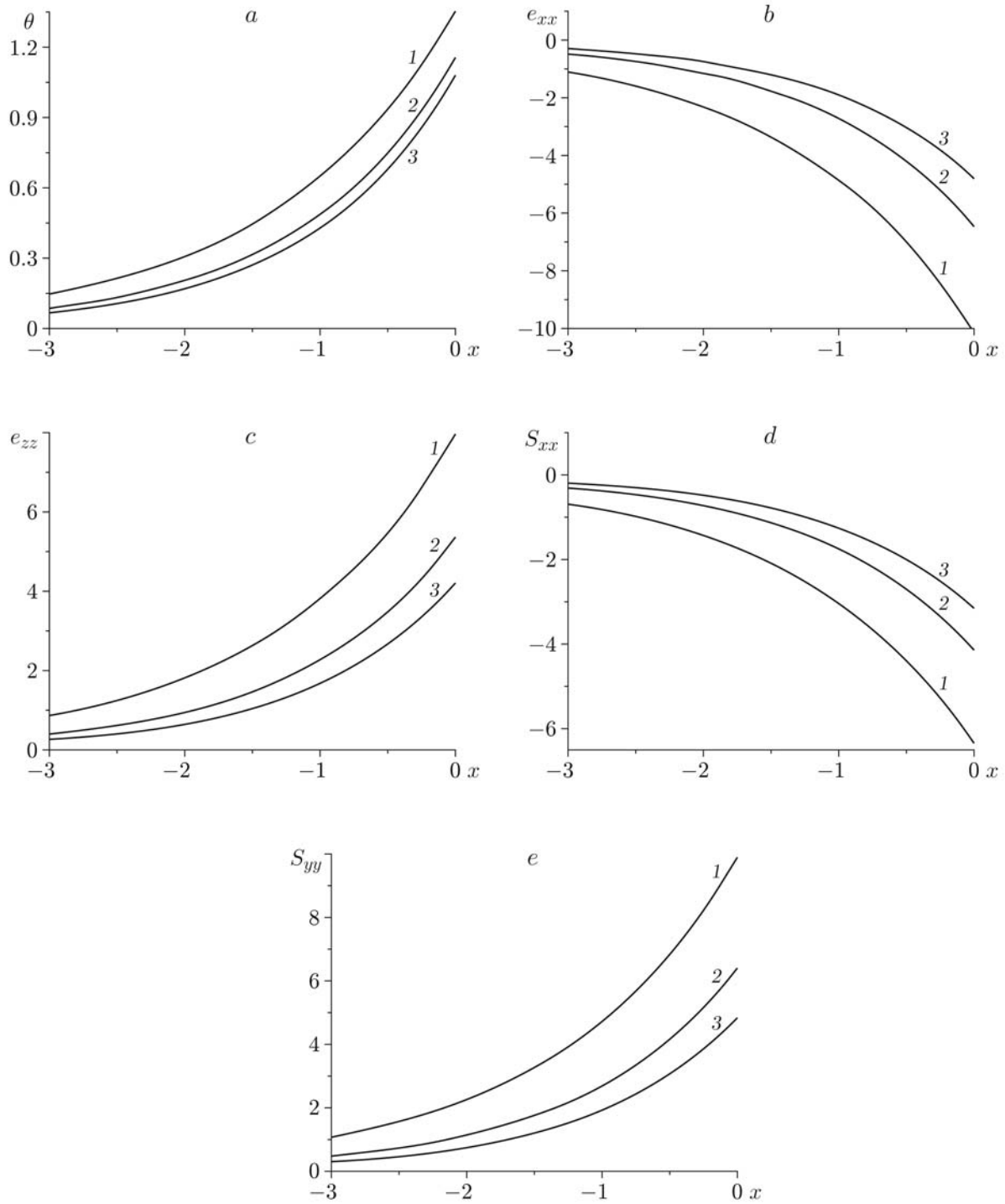
Fig. 5. Distributions of temperature (a) and non-zero components of the strain tensors (b and c) and stress tensors (d and e) in the heating zone at $\gamma_* < \alpha < \sqrt{2\gamma_2}$ ($\omega = 0.1$, $\sigma = 0.25$, and $\gamma_2 = 0.35$): $\alpha = -0.79$ (1), -0.8 (2), and -0.81 (3).

Conclusions. Thus, a model of self-sustained synthesis of a coating on a substrate is proposed. An analysis of the stationary solution shows that various synthesis regimes characterized by different reaction rates are possible even in the linearized formulation: slow regime of solid-phase combustion with the temperature of the reaction products being lower than the temperature of the reaction products in the combustion wave with ignored stresses and strains, intermediate regimes with high rates but different structures of the reaction zone, and synthesis regimes with the reaction rate higher than the velocity of propagation of mechanical perturbations in the initial substance. The issue of stability of these conversion regimes is still open and requires additional research.

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